Domain colouring

Vahagn Aslanyan

University of Manchester
https://vahagn-aslanyan.github.io
https://vahagnaslanyan.wordpress.com
vahagn.aslanyan@manchester.ac.uk

5 Dec 2025

Outline of the talk

The talk is about *domain colouring* – a technique for visualising complex functions.

Outline of the talk

The talk is about *domain colouring* – a technique for visualising complex functions.

- Visualising functions of a real variable via graphs.

Outline of the talk

The talk is about *domain colouring* – a technique for visualising complex functions.

- Visualising functions of a real variable via graphs.
- A quick introduction to complex numbers.

Outline of the talk

The talk is about *domain colouring* – a technique for visualising complex functions.

- Visualising functions of a real variable via graphs.
- A quick introduction to complex numbers.
- Assigning colours to complex numbers, and using this to visualise functions of a complex variable.

Outline of the talk

The talk is about *domain colouring* – a technique for visualising complex functions.

- Visualising functions of a real variable via graphs.
- A quick introduction to complex numbers.
- Assigning colours to complex numbers, and using this to visualise functions of a complex variable.
- Examples.

Outline of the talk

The talk is about *domain colouring* – a technique for visualising complex functions.

- Visualising functions of a real variable via graphs.
- A quick introduction to complex numbers.
- Assigning colours to complex numbers, and using this to visualise functions of a complex variable.
- Examples.
- Quiz.

Example of domain colouring

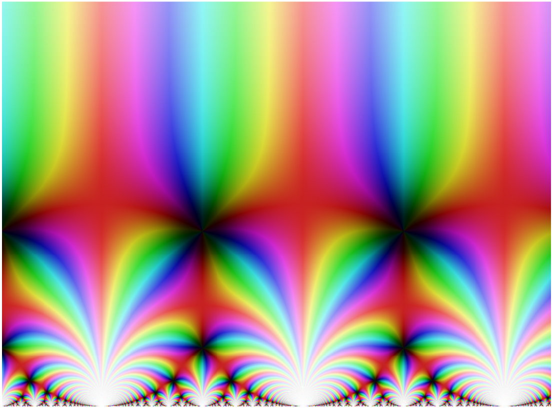
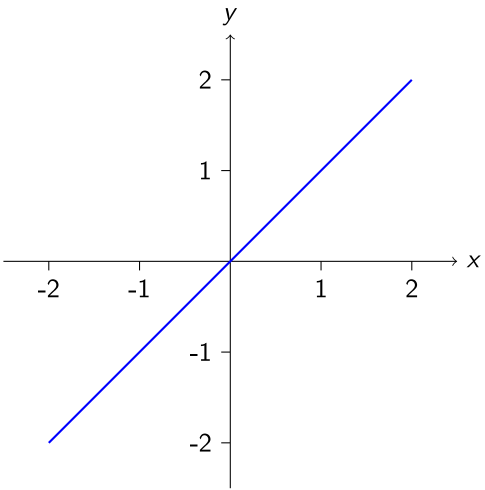
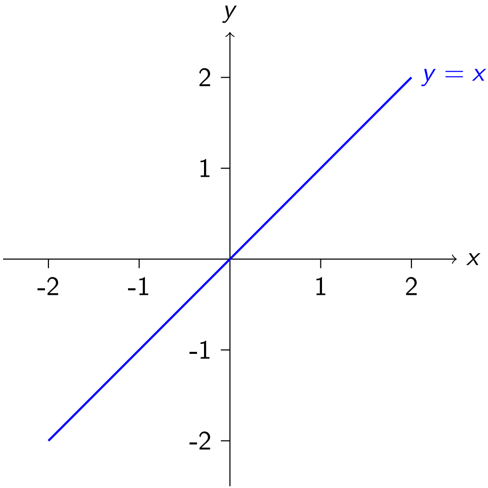


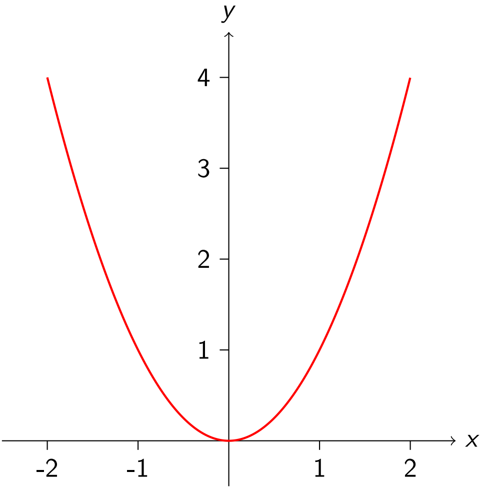
Figure: $j(z)$ (from Wikimedia Commons)

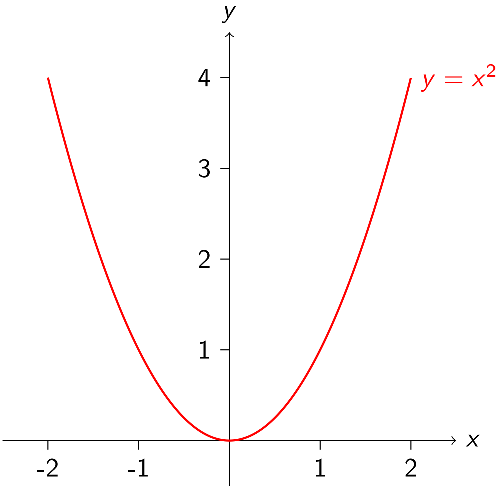
Graphs of real functions

Functions of a real variable (i.e. defined on the real line) can be visualised geometrically by their graphs.


y
2
1
x
-2
-1
1
2
-1
-2

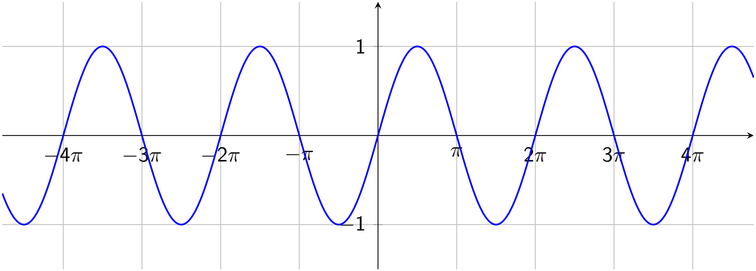

y
x
y = x
2
1
-1
-2
-2
-1
1
2


y
4
3
2
1
x
-2
-1
1
2

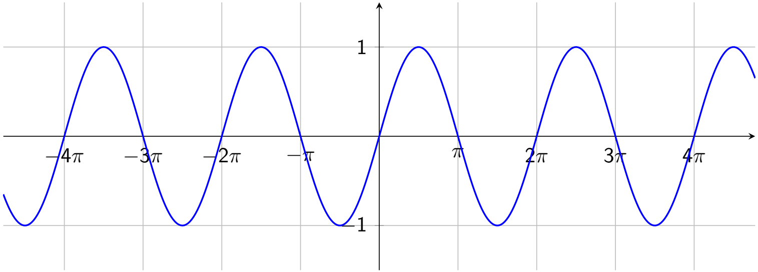

y
x
$y = x^2$
4
3
2
1
-2
-1
1
2

We can get some information about the function from the graph.

We can get some information about the function from the graph.



We can get some information about the function from the graph.



Graphs are a useful way of visualising functions.

Complex numbers

- Complex numbers are numbers of the form $c = a + b \cdot i$ where $a, b \in \mathbb{R}$ are real numbers, and $i = \sqrt{-1}$ (i.e. $i^2 = -1$).

Complex numbers

- Complex numbers are numbers of the form $c = a + b \cdot i$ where $a, b \in \mathbb{R}$ are real numbers, and $i = \sqrt{-1}$ (i.e. $i^2 = -1$).
- a is the *real part* of c and b is its *imaginary part*.

Complex numbers

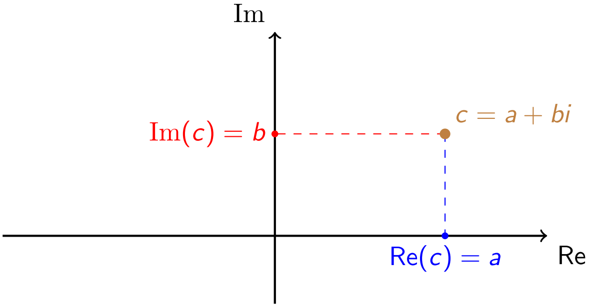
- Complex numbers are numbers of the form $c = a + b \cdot i$ where $a, b \in \mathbb{R}$ are real numbers, and $i = \sqrt{-1}$ (i.e. $i^2 = -1$).
- a is the *real part* of c and b is its *imaginary part*.
- We can identify c with the pair (a, b). So the set of complex numbers can be thought of as the real plane.

Complex numbers

- Complex numbers are numbers of the form $c = a + b \cdot i$ where $a, b \in \mathbb{R}$ are real numbers, and $i = \sqrt{-1}$ (i.e. $i^2 = -1$).
- a is the *real part* of c and b is its *imaginary part*.
- We can identify c with the pair (a, b). So the set of complex numbers can be thought of as the real plane.

Complex numbers

- Complex numbers are numbers of the form $c = a + b \cdot i$ where $a, b \in \mathbb{R}$ are real numbers, and $i = \sqrt{-1}$ (i.e. $i^2 = -1$).
- *a* is the *real part* of *c* and *b* is its *imaginary part*.
- We can identify *c* with the pair (a, b). So the set of complex numbers can be thought of as the real plane.



Adding and multiplying complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$, where $a, b, c, d \in \mathbb{R}$.

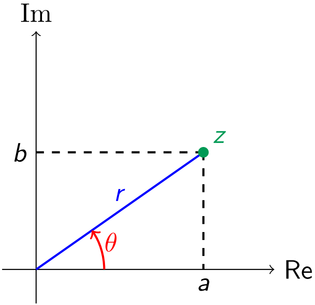
$$\text{Addition:} \quad \begin{aligned} z_1 + z_2 &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i. \end{aligned}$$

$$\text{Multiplication:} \quad \begin{aligned} z_1 \cdot z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

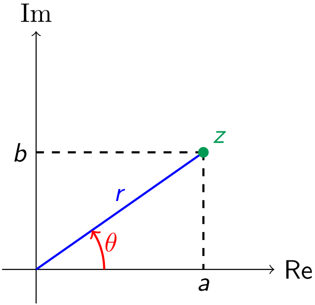
Polar coordinates

Let

$$z = a + bi.$$



Polar coordinates

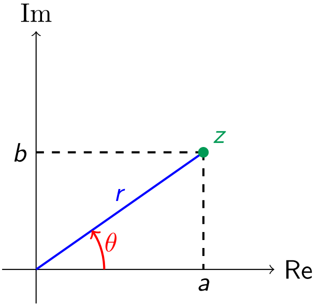


Let

$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value*

Polar coordinates

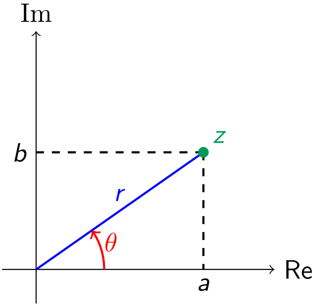


Let

$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and
$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z.

Polar coordinates



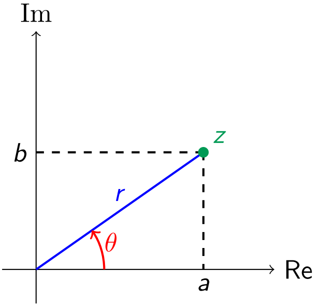
Let

$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and
$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z. Then

$$a =$$

Polar coordinates



Let

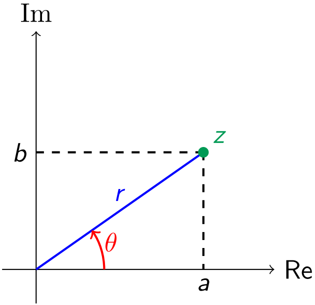
$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and

$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z. Then

$$a = r\cos\theta, \ \ b =$$

Polar coordinates



Let

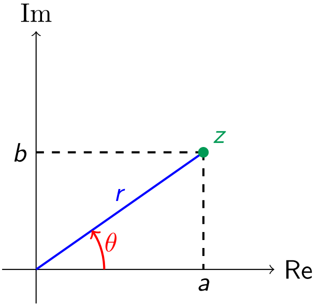
$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and

$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z. Then

$$a = r\cos\theta, \;\; b = r\sin\theta.$$

Polar coordinates



Let

$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and

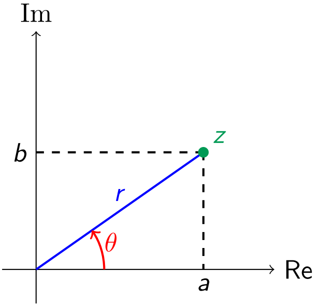
$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z. Then

$$a = r\cos\theta, \ \ b = r\sin\theta.$$

In polar form:

$$z = r(\cos\theta + i\sin\theta).$$

Polar coordinates



Let

$$z = a + bi.$$

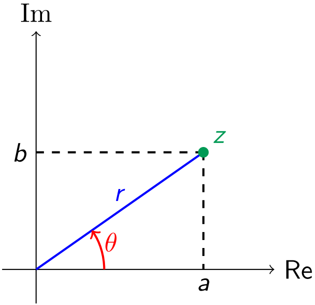
$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and
$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z. Then

$$a = r\cos\theta, \;\; b = r\sin\theta.$$

In polar form:

$$z = r(\cos\theta + i\sin\theta).$$

Define $e^{i\theta} := \cos\theta + i\sin\theta$, so $z = re^{i\theta}$.



Let

$$z = a + bi.$$

$r = |z| = \sqrt{a^2 + b^2} \in [0, +\infty)$ is the *absolute value* and
$\theta = \arg(z) \in [0, 2\pi)$ is the *argument* of z. Then

$$a = r\cos\theta, \ \ b = r\sin\theta.$$

In polar form:

$$z = r(\cos\theta + i\sin\theta).$$

Define $e^{i\theta} := \cos\theta + i\sin\theta$, so $z = re^{i\theta}$.

Multiplication in polar form:

$$z_1 = r_1 e^{i\theta_1}, \ z_2 = r_2 e^{i\theta_2} \implies z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}.$$

Side note

Using Euler's identity to derive the cosine and sine sum formulas

Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$.

Consider:

$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}.$$

Compute each side:

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta),$$

$$e^{i\alpha} \cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta).$$

Expand:

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta.$$

Group real and imaginary parts:

$$\text{Real part: } \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$\text{Imaginary part: } \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Thus:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values?

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space.

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.
- Let $f : U \to \mathbb{C}$ be a complex function where U is a subset of $\mathbb{C}$. The idea is to assign a colour to each point z of U which will somehow represent the value $f(z)$.

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.
- Let $f : U \to \mathbb{C}$ be a complex function where U is a subset of $\mathbb{C}$. The idea is to assign a colour to each point z of U which will somehow represent the value $f(z)$.
- We associate colours to complex numbers using the HSV (hue, saturation, value) model.

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.
- Let $f : U \to \mathbb{C}$ be a complex function where U is a subset of $\mathbb{C}$. The idea is to assign a colour to each point z of U which will somehow represent the value $f(z)$.
- We associate colours to complex numbers using the HSV (hue, saturation, value) model. One can also use the RGB model (which can be converted to HSV).

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.
- Let $f : U \to \mathbb{C}$ be a complex function where U is a subset of $\mathbb{C}$. The idea is to assign a colour to each point z of U which will somehow represent the value $f(z)$.
- We associate colours to complex numbers using the HSV (hue, saturation, value) model. One can also use the RGB model (which can be converted to HSV).
- HSV uses three real parameters to encode a colour.

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.
- Let $f : U \to \mathbb{C}$ be a complex function where U is a subset of $\mathbb{C}$. The idea is to assign a colour to each point z of U which will somehow represent the value $f(z)$.
- We associate colours to complex numbers using the HSV (hue, saturation, value) model. One can also use the RGB model (which can be converted to HSV).
- HSV uses three real parameters to encode a colour. Given a complex number $w = re^{i\theta}$ we use the mapping

$$re^{i\theta} \mapsto \begin{cases} H = \theta \in [0, 2\pi) \\ S = 1 \in [0, 1] \\ V = \dfrac{2}{\pi}\arctan(r) \in [0, 1) \end{cases}$$

Visualising complex functions

- How can we visualise complex functions, i.e. functions which take complex numbers as input and output complex values? We can't draw a graph as it would require working in a real 4-dimensional space. So we use colours.
- Let $f : U \to \mathbb{C}$ be a complex function where U is a subset of $\mathbb{C}$. The idea is to assign a colour to each point z of U which will somehow represent the value $f(z)$.
- We associate colours to complex numbers using the HSV (hue, saturation, value) model. One can also use the RGB model (which can be converted to HSV).
- HSV uses three real parameters to encode a colour. Given a complex number $w = re^{i\theta}$ we use the mapping

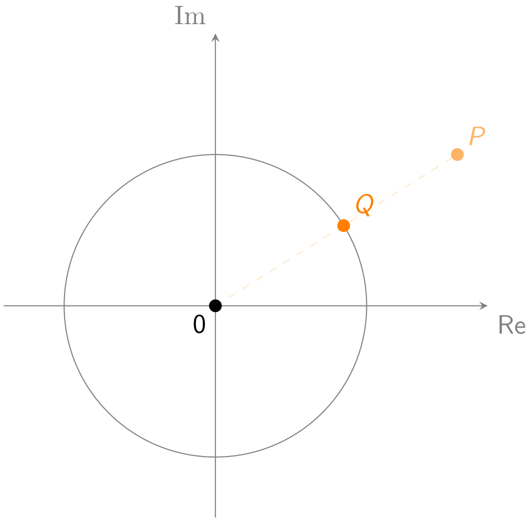
$$re^{i\theta} \mapsto \begin{cases} H = \theta \in [0, 2\pi) \\ S = 1 \in [0, 1] \\ V = \dfrac{2}{\pi} \arctan(r) \in [0, 1) \end{cases}$$

- $V = 0$ $(r = 0)$ corresponds to **Black** and $V = 1$ $(r = \infty)$ corresponds to

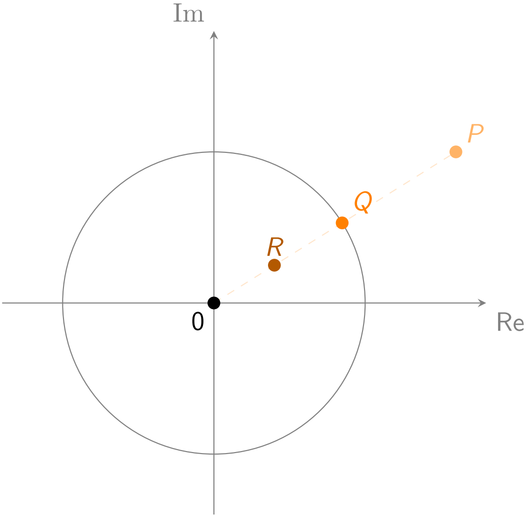
Colour wheel

Figure: HSV Colour Wheel (from Wikimedia Commons)

Brightness



Brightness



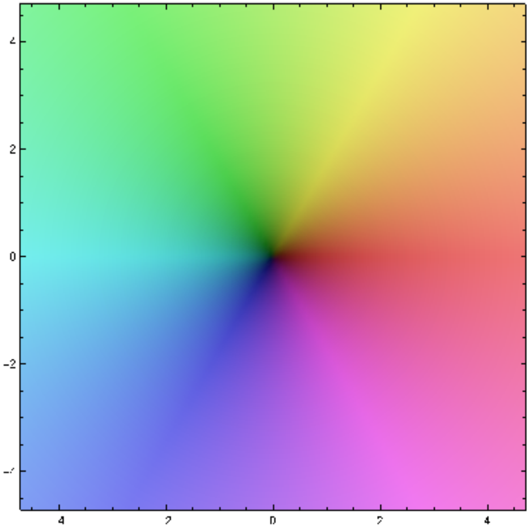


Figure: $f(z) = z$ (from WolframAlpha)

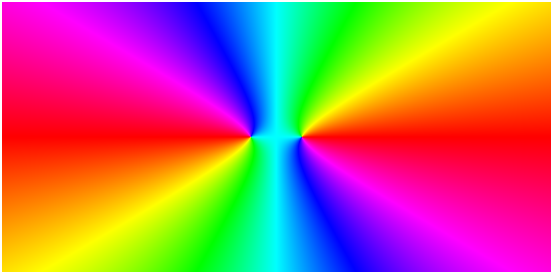


Figure: $f(z) = z^2 - 1$ (from dynamicmath.xyz)

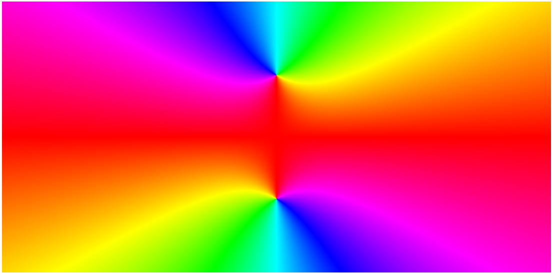


Figure: $f(z) = z^2 + 1$ (from dynamicmath.xyz)

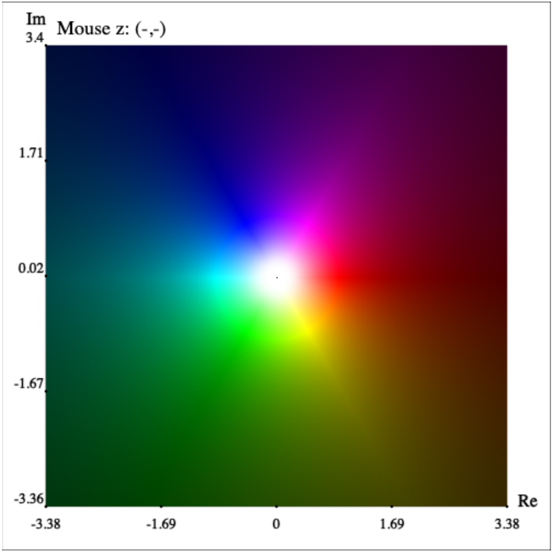


Figure: $f(z) = 1/z$ (from dynamicmath.xyz)

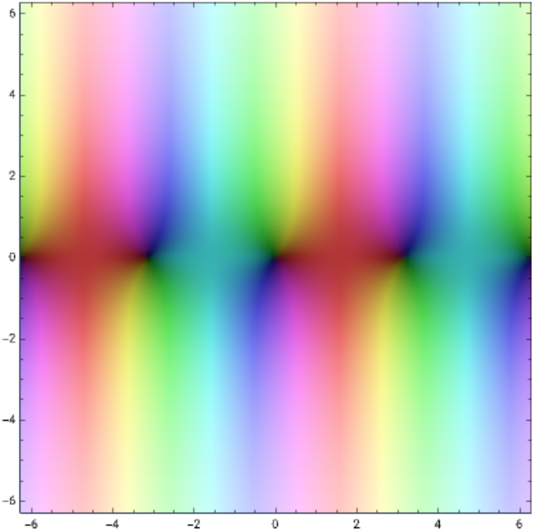


Figure: $f(z) = \sin(z)$ (from WolframAlpha)

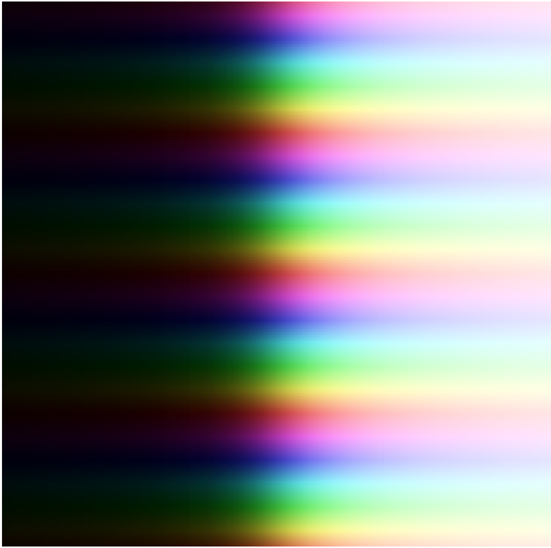


Figure: $f(z) = e^z = e^x(\cos y + i \sin y)$ (from WolframAlpha)

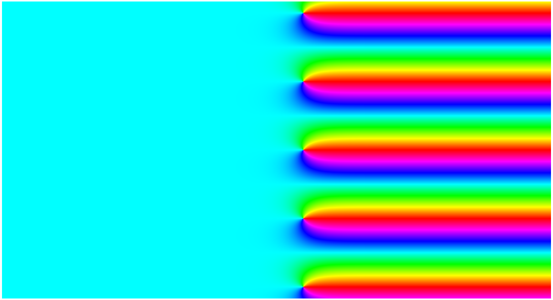


Figure: $f(z) = e^z - 1$ (from dynamicmath.xyz)

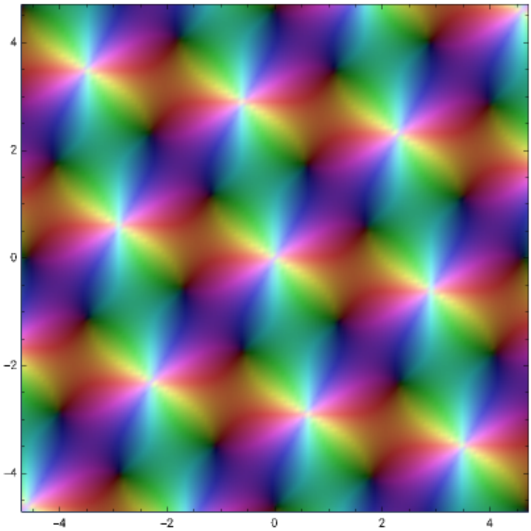


Figure: $f(z) = \wp(z)$ (from WolframAlpha)

Figure: $f(z) = \sqrt{z}$ (from dynamicmath.xyz)

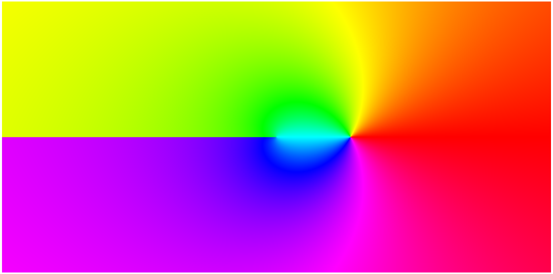


Figure: $f(z) = \log(z)$ (from dynamicmath.xyz)

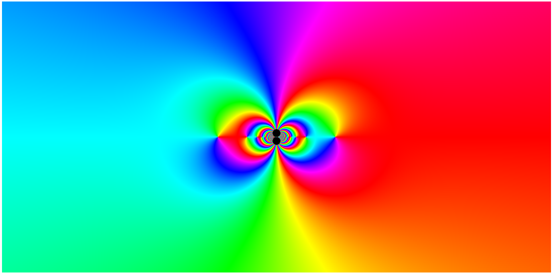


Figure: $f(z) = \sin(1/z)$ (from dynamicmath.xyz)

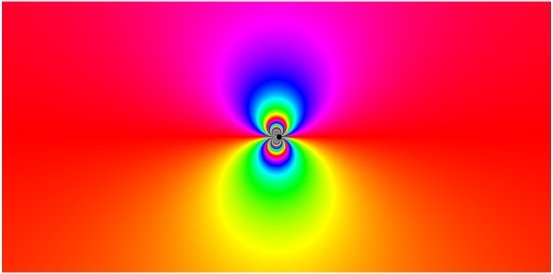


Figure: $f(z) = e^{1/z}$ (from dynamicmath.xyz)

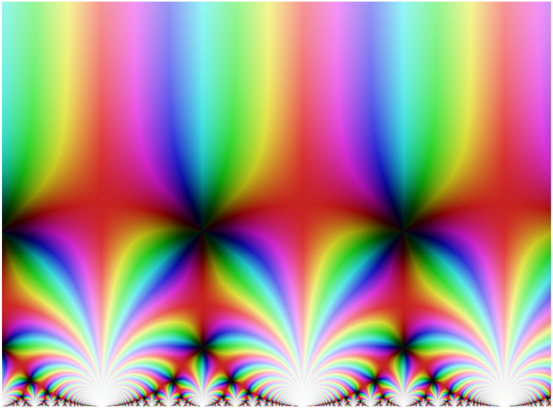


Figure: $j(z)$ (from Wikimedia Commons)

Quiz

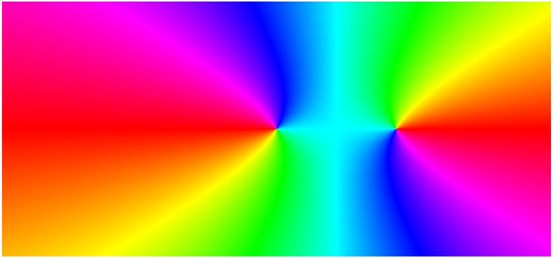


Figure: (a) z^2 (b) $1/z$ (c) $z(z-2)$

Quiz

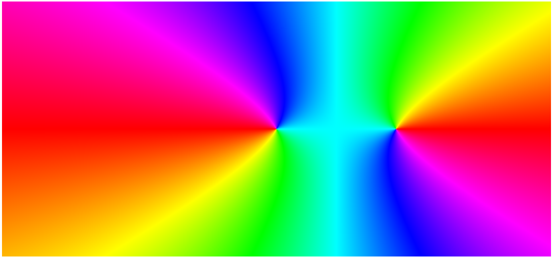


Figure: (a) z^2 (b) $1/z$ (c) $z(z-2)$

Quiz

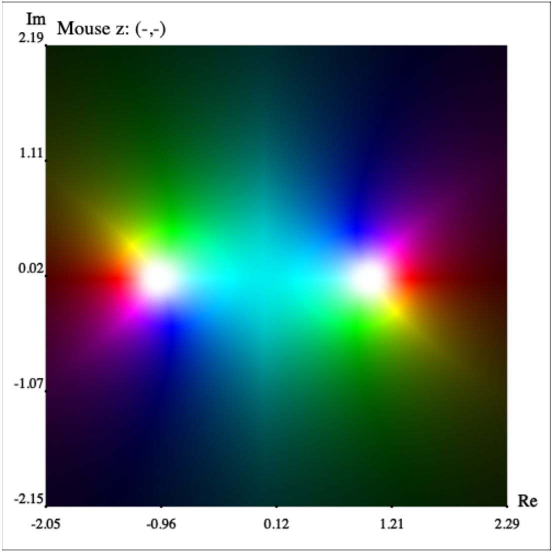


Figure: (a) $1/(z-3)$ (b) $1/(z^2-1)$ (c) z^2-1

Quiz

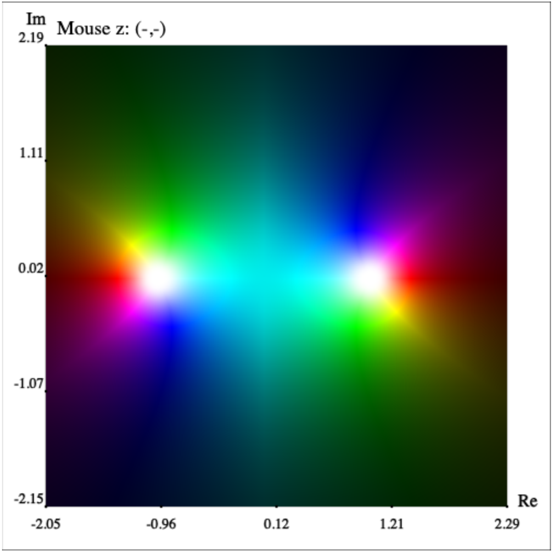


Figure: (a) $1/(z-3)$ (b) $1/(z^2-1)$ (c) z^2-1

Quiz

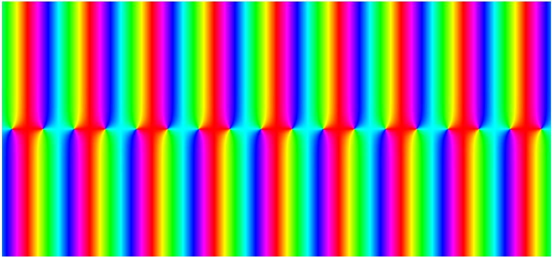


Figure: (a) $\cos(z)$ (b) z^2 (c) $\tan(z)$

Quiz

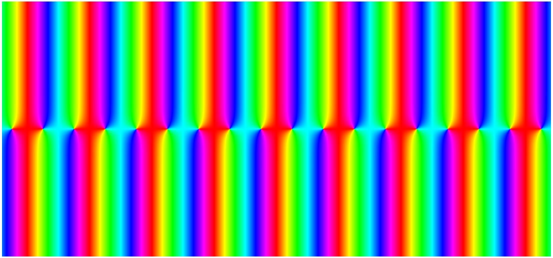


Figure: (a) $\cos(z)$ (b) z^2 (c) $\tan(z)$

Quiz

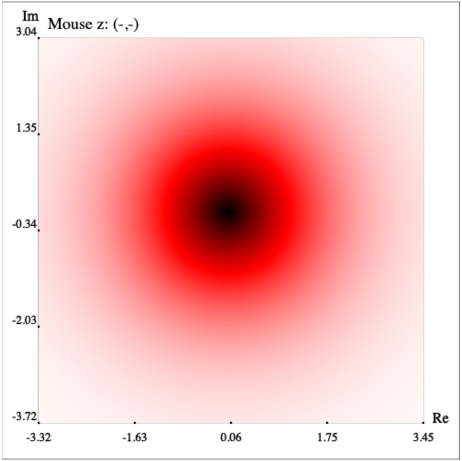


Figure: (a) 1 (b) $|z|$ (c) $\mathrm{Re}(z)$

Quiz

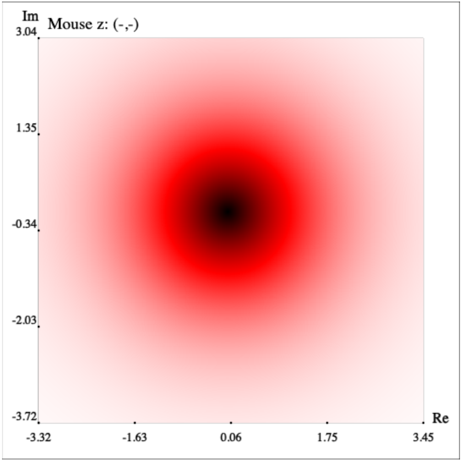


Figure: (a) 1 (b) $|z|$ (c) $\mathrm{Re}(z)$

Quiz

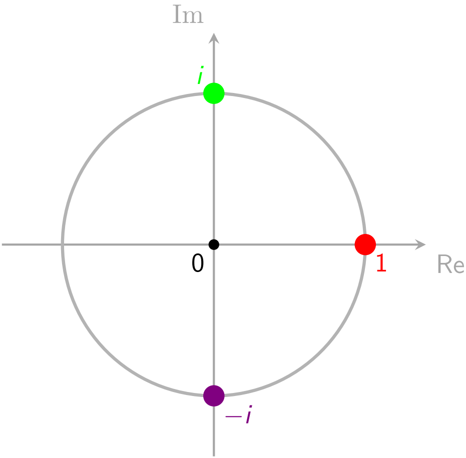
Quiz

Figure: (a) i (b) $-i$ (c) 1

Colour wheel

Figure: HSV Colour Wheel (from Wikimedia commons)

Unit circle



Quiz

Figure: (a) i (b) $-i$ (c) 1

Dynamic Mathematics website

https://www.dynamicmath.xyz/complex/function-plotter/hsv.htm?expression=ZV56LXo=